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A NUMERICAL APPROACH FOR SOLVING BAGELY-TORVIK AND FRACTIONAL OSCILLATION EQUATIONS

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Abstract. In this article, we obtain numerical solutions of Bagely-Torvik and a class of fractional oscillation equations by using a numerical method based on Hosoya and Clique polynomials. The fractional derivative is in the Coputo sense. In this method, first we convert the given fractional order differential equations to corresponding fractional integral equations, and then we use the Rayleigh-Ritz method and collocation points to transform the fractional integral equation into a system of algebraic equations. Finally, we gain a numerical result by solving the consequent algebraic system.
The Hosoya polynomial of simple paths and the Clique polynomial of completed graphs are used as basic functions. A few test examples are presented to show the efficiency of the presented method.

Keywords: Fractional calculus, Bagely-Torvik equation, Hosoya polynomials, Clique polynomial, Caputo derivative, Fractional oscillation equation, Numerical solution.
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1 Introduction

In the last few decades, fractional-order calculus has been studied as an alternative calculus in mathematics as well as in other fields of science and engineering. Fractional derivatives have been used to simulate a wide range of problems in physics, chemistry, biology, and engineering (Podlubny (1998); Ross (1977); Torvik & Bagley (1984, 1985)). Because most fractional order differential equations do not have exact analytic solutions, approximation and numerical approaches are widely used.

The fractional derivative (FD) is useful in presenting long-memory processes and materials, anomalous diffusion, long-range interactions, long-term behaviours, power laws, and so on. Because the FD has a non-local feature, it is difficult to solve fractional differential equations (FDE). Because most fractional order differential equations do not have exact analytic solutions, approximation and numerical approaches are widely used. Researchers are now working on a solution to FDE by creating new analytical and numerical methodologies. The widespread use of FDE motivates the development of analytical and numerical techniques to solve it (Narsale et al. (2023)).

One of the well-known fractional order differential equations is the Bagely-Torvik equation (BTE). In 1983, the BTE was introduced by Bagley and Torvik as an application of fractional

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calculus for studying viscoelastically damped structures and their vital role in applied science and engineering problems, especially any linearly damped fractional (Fazli & Nieto (2019); Torvik & Bagley (1984, 1985)). A general form of BTE is given as:

$$\chi\, u''(\xi) + \gamma\, D^{\frac{3}{2}} u(\xi) + \varkappa\, u(\xi) = f(\xi), \quad 0 < \xi, \qquad (1)$$
$$u(0) = u_0, \quad u'(0) = u_1,$$

where $\chi \neq 0$, γ and $\varkappa$ are real numbers. $D^{\frac{3}{2}}$is a Caputo fractional derivative and $f(\xi)$ is known function.
The BTE has been studied by many researchers.

The BTE has been solved by many methods, such as the Adomian decomposition method (Daftardar & Jafari (2005, 2007)), the variational iteration method (VIM), the fractional iteration method (FIM) (Mekkaoui & Hammouch (2012)), the shifted Legendre-collocation (SLC) (El-Gamel & El-Hady (2017)) and the Ritz method (Firoozjaee et al. (2015)). In (Fazli & Nieto (2019)), the authors proved the existence and uniqueness of solutions for the Bagely-Torvik equation.

The oscillation of a function on an interval in its domain is the difference between its extreme values, i.e. supremum and infimum. The fractional type of oscillation differential equation is (Bartusek & Dosla (2023))

$$D^{\alpha} u(\xi) + p(\xi)\, K(u(\xi)) = 0, \quad \xi > 0, \qquad (2)$$

where $n-1 < \alpha < n, \quad n \in N, \quad n > 2$, p is a real-valued positive continuous function on the interval $(0, \infty)$ so that $p \in L^1(0,1)$, and D^{α} is fractional differential operator, and the function $K \in C^0(\mathbb{R}), \quad K(\nu)\nu > 0$ for $\nu \neq 0$.

A solution u of (2) is called oscillatory if it has arbitrary zeros; otherwise, it is nonoscillatory (Bartusek & Dosla (2023)).

In this paper, we solve equations (1) and (2) by using the Rayleigh-Ritz method based on Hosoya and Clique polynomials.
The paper is organized as follows: In Section 2 basic definitions are given. Section 3 deals with presented method. Some test examples have been presented in Section 4. This is followed by the conclusions, which are summarized in Section 5.

2 Primary definitions

In this part, we first give a brief review of some basic definitions of fractional calculus, Hossoya and Clique polynomials.

2.1 Basic Definitions of FC

Definition 1. *The Riemann-Liouville fractional integral of the order ς of a function f is defined as follows (Podlubny (1998)):*
-The left-side Riemann-Liouville integral

$${}^{RL}I^{\varsigma}_{a^+}[f(\xi)] = \frac{1}{\Gamma(\varsigma)} \int_a^{\xi} (\xi - t)^{\varsigma-1} f(t)dt, \quad \xi \geq a, \;\; n-1 < \varsigma \leq n, \;\; n \in \mathbb{N}. \qquad (3)$$

-The right-side Riemann-Liouville integral

$${}^{RL}I^{\varsigma}_{b^-}[f(\xi)] = \frac{1}{\Gamma(\varsigma)} \int_{\xi}^{b} (t - \xi)^{\varsigma-1} f(t)dt, \quad \xi \leq b.$$

Definition 2. *The Riemann-Liouville fractional derivative of order ς is defined as follows (Podlubny (1998)):*
-left-side Riemann-liouville derivative

$$^{RL}D^{\varsigma}_{a^+}[f(\xi)] = \frac{1}{\Gamma(n-\varsigma)}\frac{d^n}{d\xi^n}\int_a^{\xi}(\xi-t)^{n-\varsigma-1}f(t)dt, \quad \xi \geq a.$$

- right-side Riemann-Liouville derivative

$$^{RL}D^{\varsigma}_{b^-}[f(\xi)] = \frac{(-1)^n}{\Gamma(n-\varsigma)}\frac{d^n}{d\xi^n}\int_{\xi}^{b}(t-\xi)^{n-\varsigma-1}f(t)dt, \quad \xi \leq b.$$

Definition 3. *Let $f \in C^n_{-1}, n \in \mathbb{N}$. The Caputo fractional derivative of order ς is defined as follows (Podlubny (1998)):*

-The left-side Caputo derivative

$$^{C}D^{\varsigma}_{a^+}[f(\xi)] = \frac{1}{\Gamma(n-\varsigma)}\int_a^{\xi}(\xi-t)^{n-\varsigma-1}\frac{d^n}{dt^n}f(t)dt, \quad \xi \geq a.$$

- The right-side Caputo derivative

$$^{C}D^{\varsigma}_{b^-}[f(\xi)] = \frac{(-1)^n}{\Gamma(n-\varsigma)}\int_{\xi}^{b}(t-\xi)^{n-\varsigma-1}\frac{d^n}{dt^n}f(t)dt, \quad \xi \leq b.$$

In further discussion we will denote $^{C}D^{\varsigma}_{a^+}$ and $^{RL}I^{\varsigma}_{a^+}$ as D^{ς} and I^{ς} , respectively. If the fractional derivative of the function $f(\xi)$ is integrable, then

$$I^{\varsigma}D^{\beta}f(\xi) = I^{\varsigma-\beta}f(\xi) - \sum_{r=0}^{n-1}f^{(r)}(0)\frac{(\xi-a)^{\varsigma-\beta+r}}{\Gamma(\varsigma-\beta+r+1)}, \quad n-1 \leq \beta < n, \varsigma \geq \beta, a < \xi < b. \tag{4}$$

$$I^{\varsigma}D^{\varsigma}f(\xi) = f(\xi) - \sum_{r=0}^{n-1}f^{r}(0)\frac{\xi^r}{r!}, \qquad n-1 < \varsigma \leq n. \tag{5}$$

2.2 Hosoya polynomial

Let graph G contain n vertices, and X is a set of unordered pairs of distant vertices. Each pair (u, v) of vertices in set X is named an edge of G. If the vertices u and v are connected by an edge, then u and v are adjacent vertices (Ramane et al. (2017)).

let $v_1, \quad v_2, \ldots, v_n$ be the vertices of G where v_i is adjacent to $v_{i+1}, \quad i = 1, \quad 2, \ldots, n-1$. The length of a path in graph G is the number of edges. A graph G is called connected if every pair of vertices of G is joined by a path (Ramane et al. (2017)).

(For more information refer to the book (Stevanovic (2001))).

The first distance-based index is the Wiener index, which was expressed in 1947 by H. Wiener (Tratnik & Pleteršek (2017)). Hosoya polynomial of a path graph is a generating function that was presented by Hosoya (Hosoya (1998)) in 1988 and is generalised from the Wiener number (Gecmen & Celik (2021)). The Hosoya polynomial results from certain vertices pair of path graphs (Gecmen & Celik (2021)).

Definition 4. *For a connected graph G (Ramane et al. (2017)), Hosoya polynomial based on n vertex values and, a path as ρ_n is defined as follows (Gecmen & Celik (2021))*

$$H(G, \xi) = \sum_{l=0}^{n} d(G, l)\xi^l,$$

where $d(G, l)$ is the number of pairs of vertices in the graph G with distance l and ξ is the parameter (Ramane et al. (2017)).

The Wiener index is expressed as follows:

$$W(G) = H'(G, \xi),$$

where $H'(G, \xi)$ is a derivative of $H(G, \xi)$(Ramane et al. (2017)).

Hosoya polynomials are obtained and computed as fallows (Geçmen et al. (2021)

$$\begin{aligned}
H(\rho_1, \xi) &= \sum_{l=0}^{1} d(\rho_1, l)\xi^l = 1, \\
H(\rho_2, \xi) &= \sum_{l=0}^{2} d(\rho_2, l)\xi^l = 2 + \xi, \\
H(\rho_3, \xi) &= \sum_{l=0}^{3} d(\rho_3, l)]\xi^l = 3 + 2\xi + \xi^2, \qquad (6)\\
&\vdots \\
H(\rho_n, \xi) &= \sum_{l=0}^{n} d(\rho_n, l)\xi^l = n + (n-1)\xi + (n-2)\xi^2 + \cdots + (n-(n-1))\xi^{n-1}.
\end{aligned}$$

In (Ramane et al. (2017)), the Hosoya polynomial is used to obtain the numerical solution of Fredholm integral equations. In (Gecmen & Celik (2021);Geçmen et al. (2021)), the numerical solutions of Volterra-Fredholm and Volterra integral equations has been obtained using numerical methods based on Hosoya polynomial. The Hosoya polynomial of simple paths was used to obtain operational matrices to solve time fractional advection-diffusion and Kolmogorov equations (Jafari et al. (2023); Zhou et al. (2023)).

2.3 Clique polynomial

Definition 5. *The Clique polynomial (CD) of a graph G is expressed as follows:*

$$C(G, \xi) = \sum_{i=0}^{n} a_i(G)\xi^i,$$

where $a_0(G) = 1$ and $a_i(G)$ is the number of i-Cliques of graph G or the number of complete subgraphs with i vertices (Hoede & Li (1994)),(Ganji et al. (2021)). If $V(G) = 0$, we have $C(G, \xi) = 1$. $a_1 = |V(G)|$ is the number of vertices of graph G, and $a_2 = |E(G)|$ is the number of edges of graph G (Hoede & Li (1994)).

The Clique polynomial of a complete graph G with i vertices is expressed by (Ganji et al. (2021))

$$C(G_i, \xi) = (1 + \xi)^i.$$

3 The method

Consider the Bagely-Torvik equation (1) when $\chi = 1$ as follows:

$$\begin{aligned}
&u''(\xi) + \gamma D^{\frac{3}{2}} u(\xi) + \varkappa u(\xi) = f(\xi), \quad \xi > 0, \qquad (7)\\
&u(0) = u_0, \quad u'(0) = u_1.
\end{aligned}$$

To solve (7), we apply the I^2 operator on both sides of the above equation. It gives:

$$I^2 u''(\xi) + \gamma I^2 D^{\frac{3}{2}} u(\xi) + \varkappa I^2 u(\xi) = I^2 f(\xi),$$

$$u(\xi) - u_1\xi - u_0 + \gamma I^2 D^{\frac{3}{2}} u(\xi) + \varkappa I^2 u(\xi) = I^2 f(\xi), \tag{8}$$

In view of (4), equation (8) can be written as follows:

$$u(\xi) = u_1\xi + u_0 - \gamma \left(I^{\frac{1}{2}} u(\xi) - \sum_{r=0}^{1} u^r(0) \frac{(\xi - a)^{\frac{1}{2}+r}}{\Gamma(\frac{1}{2} + r + 1)} \right) - \varkappa I^2 u(\xi) + I^2 f(\xi),$$

so,

$$u(\xi) = u_1\xi + u_0 - \gamma \left(I^{\frac{1}{2}} u(\xi) - \frac{2u_0}{\sqrt{\pi}} (\xi - a)^{\frac{1}{2}} - \frac{4u_1}{3\sqrt{\pi}} (\xi - a)^{\frac{3}{2}} \right) - \varkappa I^2 u(\xi) + I^2 f(\xi).$$

We get,

$$u(\xi) = \tau(\xi) - \gamma I^{\frac{1}{2}} u(\xi) - \varkappa I^2 u(\xi), \tag{9}$$

where

$$\tau(\xi) = u_1\xi + u_0 + \gamma \left(\frac{2u_0}{\sqrt{\pi}} (\xi - a)^{\frac{1}{2}} + \frac{4u_1}{3\sqrt{\pi}} (\xi - a)^{\frac{3}{2}} \right) + I^2 f(\xi).$$

Using equation (3), we rewrite the equation (9) in the following form:

$$u(\xi) = \tau(\xi) - \gamma \frac{1}{\sqrt{\pi}} \left(\int_0^{\xi} (\xi - t)^{\frac{-1}{2}} u(t) dt \right) - \varkappa \left(\int_0^{\xi} (\xi - t) u(t) dt \right). \tag{10}$$

We use the following theorem to assume the approximate solution of (10) in a finite series form.

Theorem 1. *(Tajadodi et al. (2022)) Let* $u(\xi) \in L^2[0,1]$, $\Upsilon(\xi)$ *polynomial in the matrix form* $\Upsilon(\xi) = [\Upsilon_1(\xi), \Upsilon_2(\xi), \ldots, \Upsilon_N(\xi)]^T$ *and* C *coefficient in the matrix form* $C = [c_1, c_2, \ldots, c_N]^T$, *then function* $\Upsilon(\xi)$ *can be approximated as follows:*

$$u(\xi) = \sum_{n=0}^{N} C_i \Upsilon_i(\xi) = C^T \Upsilon(\xi).$$

According to the above theorem, function $u(\xi) \in L_2[0,1]$ can be expanded based on the Hosoya (or Clique) polynomial as fallows:

$$u(\xi) = \sum_{i=1}^{n} \lambda_i H(\rho_i, \xi) = \Lambda^T \mathbb{H}_\rho(\xi), \tag{11}$$

where Λ and $\mathbb{H}_\rho(\xi)$ are $n \times 1$ matrices,

$$\begin{aligned} \Lambda &= [\lambda_1, \lambda_2, \ldots, \lambda_n]^T, \\ \mathbb{H}_\rho(\xi) &= [H(\rho_1, \xi), H(\rho_2, \xi), \ldots, H(\rho_n, \xi)]^T. \end{aligned}$$

Now we substitute (11) in the equation (10). It leads to

$$\Lambda^T \mathbb{H}_\rho(\xi) = \tau(\xi_j) - \gamma \frac{1}{\sqrt{\pi}} \left(\int_0^{\xi} (\xi - t)^{\frac{-1}{2}} \Lambda^T \mathbb{H}_\rho(t) dt \right) - \varkappa \left(\int_0^{\xi} (\xi - t) \Lambda^T \mathbb{H}_\rho(t) dt \right),$$

$$\Lambda^T \left(\mathbb{H}_\rho(\xi) + \gamma \frac{1}{\sqrt{\pi}} \int_0^{\xi} (\xi - t)^{\frac{-1}{2}} \mathbb{H}_\rho(t) dt + \varkappa \int_0^{\xi} (\xi - t) \mathbb{H}_\rho(t) dt \right) = \tau(\xi),$$

so,

$$\Lambda^T \left(\mathbb{H}_\rho(\xi) + \phi(\xi) \right) = \tau(\xi), \tag{12}$$

where

$$\phi(\xi) = \gamma \frac{1}{\sqrt{\pi}} \left(\int_0^{\xi} (\xi - t)^{\frac{-1}{2}} \mathbb{H}_\rho(t) dt \right) + \varkappa \left(\int_0^{\xi} (\xi - t) \mathbb{H}_\rho(t) dt \right).$$

To obtain the unknown coefficient λ_i, $i = 1, 2, \cdots, n$ in (12), we use collocation points $\xi_j = \frac{j-0.5}{n}, j = 1, 2, \ldots, n$. By substituting the collocation points in (12), we will have

$$\Lambda^T \left(\mathbb{H}_\rho(\xi_j) + \phi(\xi_j)\right) = \tau(\xi_j), \tag{13}$$

where

$$\phi(\xi_j) = \gamma \frac{1}{\sqrt{\pi}} \left(\int_0^{\xi_j} (\xi_j - t)^{\frac{-1}{2}} \mathbb{H}_\rho(t) dt \right) + \varkappa \left(\int_0^{\xi_j} (\xi_j - t) \mathbb{H}_\rho(t) dt \right).$$

which is a system of algebraic equations. Finally, by solving the obtained system of algebraic equations with a mathematics software, the coefficients Λ will be computed. After that, by replacing coefficients Λ_i in equation (11), we can achieve the desired results.

Remark 1. *We can use Click polynomials or Taylor polynomials instead of Hosoya polynomials in* (11).

Remark 2. *We use the given collocation points when $\xi \in [0, 1]$ for the other interval we use change of variables.*

4 Test Examples

In this section, we solve different cases of equations (1) and (2) with the presented method in the previous section, and compare our results with obtained results by using other methods.

Example 1. *Consider the following fractional Bagely-Torvik equation:(Fazli & Nieto (2019))*

$$u''(\xi) - \tfrac{2}{5} D^{\frac{3}{2}} u(\xi) - \tfrac{1}{2} u(\xi) = f(\xi), \qquad 0 < \xi \leq 1 \tag{14}$$
$$u(0) = 0, \ u'(0) = \tfrac{9}{16},$$

where $f(\xi) = -\frac{1}{2}\xi^3 + \frac{3}{4}\xi^2 + \frac{183}{32}\xi - 3 - \frac{4}{5}\frac{\sqrt{\xi}(-3+4\xi)}{\sqrt{\pi}}$. *The exact solution is* $u(\xi) = \xi^3 - \frac{3}{2}\xi^2 + \frac{9}{16}\xi$.

By applying the inverse operator I^2 on the both sides of (14) and substituting the given initial conditions, we have

$$u(\xi) = \frac{9}{16}\xi - \frac{3\xi^{\frac{3}{2}}}{10\sqrt{\pi}} + \int_0^\xi (\xi - t) f(t) dt + \frac{2}{5\sqrt{\pi}} \int_0^\xi (\xi - t)^{\frac{-1}{2}} u(t) dt + \frac{1}{2} \int_0^\xi (\xi - t) u(t) dt.$$

By assuming $u(\xi) = \sum_{i=1}^4 \lambda_i H(\rho_i, \xi)$ and substituting it in the above equation, we obtain λ_i by solving the system of equations at the collocation point $\xi_j = \frac{j-\frac{1}{2}}{n} \quad j = 1, 2, 3, 4$ as fallows:

$$\lambda_1 = 2.625, \quad \lambda_2 = 4.5625, \quad \lambda_3 = -3.50 \text{ and } \lambda_4 = 1.$$

Then, by putting the above coefficients in $u(\xi)$, we have

$$\begin{aligned} u(\xi) = \sum_{i=1}^4 \lambda_i H(\rho_i, \xi) &= -2.625 + 4.5625(2 + \xi) - 3.5(3 + 2\xi + \xi^2) + 4 + 3\xi + 2\xi^2 + \xi^3 \\ &= \xi^3 - 1.5\xi^2 + 0.5625\xi, \end{aligned}$$

which is an approximate solution for this problem. Table 1 and figure 1 show the results of the desired method.

Example 2. *Let $\chi = \gamma = \varkappa = 1$ and $f(\xi) = 7\xi + \frac{8}{\sqrt{\pi}}\xi^{\frac{3}{2}} + \xi^3 + 1$ in the Bagely-Torvik's equation* (1) *(Mekkaoui & Hammouch (2012)):*

$$u''(\xi) + D^{\frac{3}{2}} u(\xi) + u(\xi) = f(\xi), \qquad 0 < \xi < 1, \tag{15}$$
$$u(0) = u'(0) = 1.$$

The exact solution of this problem is $u(\xi) = \xi^3 + \xi + 1$.

ξ	Approximat Sol. (Hosoya Poly.)	Approximat Sol. (Clique poly.)
0.125	0.0488281	0.0497254
0.375	0.0527344	0.0531018
0.625	0.00976563	0.0105202
0.875	0.0136719	0.0142122

Table 1: The numerical results obtained based on Hosoya and Clique polynomials **Example** 1

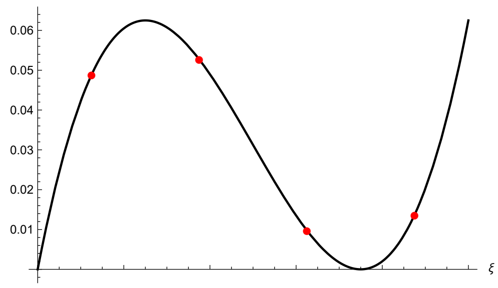

Figure 1: Comparison between the exact solution (Black Line) and approximate solution (Red Circles) based on Hosoya polynomial for **Example** 1

In similar way, we assume $u(\xi) = \sum_{i=1}^{4} \lambda_i H(\rho_i, \xi)$. Then, we substitute $u(\xi)$ into corresponding integral equation of (15). After that, we used collocation points to obtain λ_i. It leads to

$$u(\xi) = \sum_{i=1}^{4} \lambda_i H(\rho_i, \xi) = -1 + 2(2+\xi) - 2(3 + 2\xi + \xi^2) + (4 + 3\xi + 2\xi^2 + \xi^3) = \xi^3 + \xi + 1.$$

which is the exact solution.

We compared the approximate results based on the Hosoya polynomial with the obtained results by the variational iteration method (VIM) (Mekkaoui & Hammouch (2012)), the fractional iteration method (FIM) Mekkaoui & Hammouch (2012), and the shifted Legendre-collocation (SLC) (El-Gamel & El-Hady (2017)) in Table 2 .

ξ	Approximat Sol. (Hosoya Poly.)	VIM	FIM	SLC
0.10	1.101000	1.183140	1.103763	1.101000
0.25	1.265625	1.438783	1.269040	1.265625
0.50	1.625000	1.519844	1.623997	1.625000
0.75	2.171875	0.83.835	2.166900	2.171875
1.00	3.000000	-1.113593	2.994988	3.000002

Table 2: Comparison approximate solution based on the Hosoya polynomial with VIM, FIM and SLC for **Example** 2

Example 3. *Consider the following BT equation:(Daftardar & Jafari (2007))*

$$u''(\xi) + D^{\frac{3}{2}} u(\xi) + u(\xi) = 1 + \xi,$$
$$u(0) = u'(0) = 1,$$

where the exact solution is $u(\xi) = \xi + 1$.

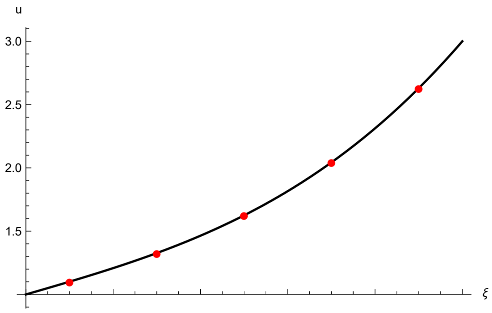

Figure 2: Comparison between the exact solution (Black Line) and approximate solution (Red Circles) for **Example** 2

We assume $u(\xi) = \sum_{i=1}^{2} \lambda_i H(\rho_i, \xi)$. Then, by applying same procedure, we obtain $\lambda_1 = -1$ and $\lambda_2 = 1$. So $u(\xi) = -1 + (2 + \xi) = 1 + \xi$, which is the exact solution.

Example 4. *Consider the following type of homogenous Bagely-Torvik's equation (Daftardar & Jafari (2005)):*

$$u^2(\xi) + u^{\alpha_1}(\xi) + u(\xi) = 0,$$
$$u(0) = 1, \quad u'(0) = 0,$$

where $\alpha \in (0,1), \alpha_1 = \alpha + 1$. *In (Daftardar & Jafari (2005)), the authors solved it using the Adomian decomposition method. They obtained* $u(\xi) = \sum_{m=0}^{\infty} \left(\sum_{j=0}^{[\frac{m}{2}]} \left(\frac{a_{mj}(-1)^{m+j} \xi^{m(1-\alpha)+2j\alpha}}{\Gamma(m(1-\alpha)+2j\alpha+1)} \right) \right)$,

where

$$a_{mj} = \begin{cases} a_{m-1,j} + a_{m-2,j-1} & ,1 \le j \le \frac{m}{2} \\ 1 & ,m = j = 0 \\ 0 & ,\text{otherwise} \end{cases}$$

The obtained numerical result when $n = 20$ in (11) based on Hosoya polynomials and the given result by the Adomian decomposition method for $m = 20$ are compared in Table 3 and Figure3.

ξ	Approximate Sol. (Hosoya Poly.)	The ADM
0.15625	0.98906	0.989066
0.46875	0.914628	0.914631
0.78125	0.790792	0.790795
1.09375	0.636988	0.63699
1.40625	0.469136	0.469137
1.71875	0.300289	0.30029
2.03125	0.14078	0.140781
2.34375	-0.00167922	0.00167898
2.65625	-0.121861	-0.121861
2.96875	-0.216802	-0.216803
3.28125	-0.285527	-0.285528
3.59375	-0.328721	-0.328722
3.90625	-0.348382	-0.348384
4.21875	-0.347464	-0.347467
4.53125	-0.32954	-0.329544
4.84375	-0.29849	-0.298498

Table 3: Approximate solutions based on the Hosoya polynomial and the ADM (Daftardar & Jafari (2007)) when $\alpha = 0.25$ in the interval $[0, 5]$ for **Example** 4

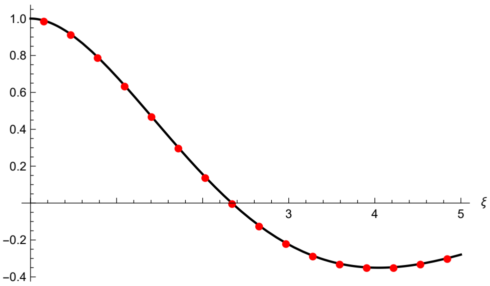

Figure 3: Comparison between obtained result with presented method (Red Circles) and the ADM (Black Line) for **Example** 4

Example 5. *Consider the following fractional oscillation equation:(Daftardar & Jafari (2005))*

$$\begin{aligned} &D^{(1+\alpha)}u(\xi) + b\,u(\xi) = f(\xi), \qquad \alpha \in (0,1], \\ &u(0) = u_0, \quad u'(0) = u_1. \end{aligned} \tag{16}$$

To solve the above equation, we apply $I^{(1+\alpha)}$ on the both sides of (16).

$$I^{(1+\alpha)}D^{(1+\alpha)}u(\xi) + bI^{(1+\alpha)}u(\xi) = I^{(1+\alpha)}f(\xi).$$

In view of (3),(5) and the given initial condition, the previous equation can be rewritten as follows:

$$u(\xi) + \left(\frac{b}{\Gamma(1+\alpha)}\int_0^{\xi}(\xi-t)^{\alpha}u(t)dt\right) = \tau(\xi), \tag{17}$$

where

$$\tau(\xi) = u_0 + \xi u_1 + \frac{b}{\Gamma(1+\alpha)}\int_0^{\xi}(\xi-t)^{\alpha}f(t)dt.$$

We assume $u(\xi)$ as fallows

$$u(\xi) = \sum_{i=1}^{n}\lambda_i H(\rho_i,\xi) = \Lambda^T\mathbb{H}_{\rho}(\xi), \tag{18}$$

where Λ is a vector of unknown coefficient and $H(\rho_i,\xi)$ are the Hosoya polynomial of simple paths.

By putting equation (18) in the equation (17) and using collocation points $\xi_j = \frac{j-0.5}{n}, j = 1,2,\ldots,n$, we get

$$\Lambda^T\mathbb{H}_{\rho}(\xi_j) + \left(\frac{b}{\Gamma(1+\alpha)}\int_0^{\xi_j}(\xi_j-t)^{\alpha}\Lambda^T\mathbb{H}_{\rho}(t)dt\right) = \tau(\xi_j),$$
$$\Lambda^T\left(\mathbb{H}_{\rho}(\xi_j) + \frac{b}{\Gamma(1+\alpha)}\int_0^{\xi_j}(\xi_j-t)^{\alpha}\mathbb{H}_{\rho}(t)dt\right) = \tau(\xi_j),$$

so

$$\Lambda^T\left(\mathbb{H}_{\rho}(\xi_j) + \phi(\xi_j)\right) = \tau(\xi_j), \tag{19}$$

where

$$\phi(\xi_j) = \frac{b}{\Gamma(1+\alpha)}\int_0^{\xi_j}(\xi_j-t)^{\alpha}\mathbb{H}_{\rho}(t)dt.$$

Finally, by solving the system of equations (19), the coefficients Λ are obtained, and by replacing the coefficients Λ in equation (18), we can achieve an approximate solution.

Let $f(\xi) = 0$ and $b = 1$, the approximate solution of (19) by using the Adomian decomposition method is $u(\xi) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \xi^{k(1+\alpha)}}{\Gamma(1+k(1+\alpha))}$(Daftardar & Jafari (2005)). In Table 4 and Figure4 we compared the numerical result obtained by the presented method based on Hosoya polynomials when $n = 11$ in (11) with the given result by the Adomian decomposition method (Daftardar & Jafari (2005)).

ξ	Approximate Sol. (Hosoya Poly.)	Approximate Sol. (Clique Poly.)	ADM (Daftardar & Jafari (2005))
0.363636	0.928159	0.927047	0.928138
1.09091	0.444878	0.442113	0.444827
1.81818	-0.237789	-0.24011	-0.237836
2.54545	-0.768932	-0.771244	-0.768953
3.27273	-0.894509	-0.893689	-0.894496
4.	-0.575048	-0.572322	-0.575008
4.72727	0.00760887	0.0101156	0.00765285
5.45455	0.55291	0.555356	0.55245
6.18182	0.795351	0.797314	0.793956
6.90909	0.629724	0.633225	0.630508
7.63636	0.165313	0.168418	0.162068

Table 4: Comparison between the obtained result by Hosoya and Clique polynomials with ADM (Daftardar & Jafari (2005)) when $\alpha = 0.95$ in the interval $[0, 8]$ for **Example** 5

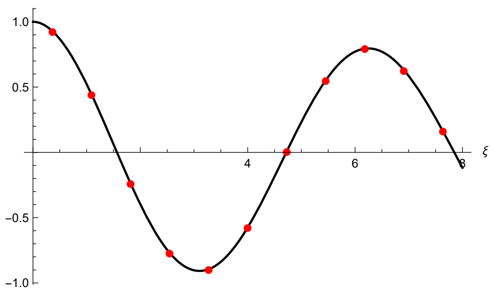

Figure 4: Comparison between approximate solution of the ADM (Black Line) and the presented method based on Hosoya polynomial (Red Circles) for **Example** 5

5 Conclusion

In this paper, we have solved Bagley-Torvik and fractional oscillation equations using a numerical methods based on Hosoya and Clique polynomials. In this method, we approximate the solution of the governing equation by a finite series with the known basis functions (Hosoya and Clique polynomials). To obtain unknown coefficients we used collocation points. The method is simple and efficient. It might be used for solving system of fractional PDE and fractional variational problems. *Mathematica* has been used for computation.

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